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ON LINEAR EQUATIONS WITH AN INFINITE NUMBER OF VARIABLES.

By Maxime Bôcher and Louis Brand.

E. Schmidt's treatment of a system of linear equations with an infinite number of variables* is of such essential simplicity and importance that it seems destined to become classical. The original memoir, however, owing to its condensation and to the rather abstract form which it has in parts is not entirely easy reading for the beginner, and Kowalewski's presentation,† while attractive in some respects, is extremely long and so arranged that unless one reads the whole it is almost impossible to get at the essential results.

The following treatment, which so far as it goes is complete in itself, is a modification of those heretofore given. Its characteristic features are, on the one hand, that it avoids altogether the process of normalization which plays such an essential and often repeated rôle in the earlier treatments; and, on the other hand, that it deals first with the case of a finite number of equations involving an infinite number of variables and regards the case of an infinite number of equations as a limit.

For the sake of clearness, though this is not logically necessary, the algebraic case of a finite number of variables is taken up first.

1. Complex Quantities with k Components. The real and complex quantities of ordinary algebra shall be termed *scalars* in distinction to the higher complex quantities, $(a_1, a_2, \cdots, a_k)$, which are aggregates of k scalars —the *components* of the complex quantity—taken in a definite order. Such complex quantities will be denoted by Greek letters. That complex quantity whose components are all zero shall be denoted by 0. Two complex quantities,

$$\alpha = (a_1, a_2, \cdots, a_k), \quad \beta = (b_1, b_2, \cdots, b_k),$$

are said to be equal when and only when $a_i = b_i \, (i = 1, 2, \cdots, k)$. We define the sum of α and β by

$$\alpha + \beta \equiv (a_1 + b_1, a_2 + b_2, \cdots, a_k + b_k);$$

and the product of α by a scalar, p, by

$$p\alpha \equiv \alpha p \equiv (pa_1, pa_2, \cdots, pa_k).$$

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* Rendiconti del Circolo Matematico di Palermo, vol. 25 (1908), pp. 56–77.

† Einführung in die Determinantentheorie (Veit: Leipzig, 1909), pp. 407–455.

The n complex quantities, $\alpha_1, \alpha_2, \cdots, \alpha_n$ are said to be linearly dependent if there exist n scalars, $c_1, c_2, \cdots, c_n$, not all zero, such that

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = 0.$$

In view of the definition of the complex quantity 0 this is equivalent to saying that $\alpha_1, \alpha_2, \cdots, \alpha_n$ are linearly dependent when and only when the n sets of k scalars each forming their components are linearly dependent. Any $k + 1$ complex quantities having k components are therefore linearly dependent.* When less than k complex quantities are given, there are always others linearly independent of them.

We also consider the *inner product*, or simply *product*, of two complex quantities α and β, defined to be the scalar

$$\alpha\beta \equiv a_1b_1 + a_2b_2 + \cdots + a_kb_k.$$

We note that $\alpha\beta$ may vanish when $\alpha \neq 0$, $\beta \neq 0$. From this definition it is clear that the commutative and distributive laws,

$$\alpha\beta = \beta\alpha, \qquad \alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma,$$

and the associative law in the case of multiplication by a scalar p,

$$p(\alpha\beta) = (p\alpha)\beta = \alpha(p\beta),$$

all hold good. The associative law, in the case of the product of three or more complex quantities, is not true. Thus $\alpha\beta\gamma$ is meaningless unless either $(\alpha\beta)\gamma$ or $\alpha(\beta\gamma)$ is specified.

A dash above a scalar shall denote, as usual, its conjugate imaginary scalar; and we shall extend this notation by writing

$$\bar{\alpha} \equiv (\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_k).$$

Then

$$\overline{\alpha\beta} = \bar{\alpha}\bar{\beta}.$$

By the *norm* of the complex quantity α is understood the scalar

$$\text{norm } \alpha \equiv \alpha\bar{\alpha} = a_1\bar{a}_1 + a_2\bar{a}_2 + \cdots + a_k\bar{a}_k = |a_1|^2 + |a_2|^2 + \cdots + |a_k|^2,$$

which is always real. Clearly norm α = norm $\bar{\alpha}$. Norm α is 0 when and only when $\alpha = 0$, and is otherwise positive.

2. Homogeneous Linear Algebraic Equations. Consider now a system of n homogeneous equations in k unknowns

$$\left\{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1k}x_k = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2k}x_k = 0 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kk}x_k = 0. \end{array}\right.$$

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* See, for example, Bôcher's Higher Algebra, § 13.

We may regard the coefficients of each of these equations as the components of a complex quantity:

$$\alpha_i = (a_{i1}, a_{i2}, \cdots, a_{ik}) \qquad (i = 1, 2, \cdots, n),$$

and also the x's as the components of the complex quantity

$$\xi = (x_1, x_2, \cdots, x_k).$$

Our system of equations may then be written

$$\alpha_1\xi = 0,\ \alpha_2\xi = 0,\ \cdots,\ \alpha_n\xi = 0. \tag{1}$$

THEOREM 1. *If* ξ *satisfies equations* (1) *and is linearly dependent upon* $\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_n$, *then* $\xi = 0$.

For suppose that

$$\xi = c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + \cdots + c_n\bar{\alpha}_n.$$

Then multiplying equations (1) by $\bar{c}_1, \bar{c}_2, \cdots, \bar{c}_n$ respectively and adding we get

$$(\bar{c}_1\alpha_1 + \bar{c}_2\alpha_2 + \cdots + \bar{c}_n\alpha_n)\xi = \bar{\xi}\xi = 0.$$

Hence $\xi = 0$, as was to be proved.

COROLLARY. If ξ satisfies the equations

$$\bar{\alpha}_1\xi = 0,\ \bar{\alpha}_2\xi = 0,\ \cdots,\ \bar{\alpha}_n\xi = 0$$

and is linearly dependent upon $\alpha_1, \alpha_2, \cdots, \alpha_n$, then $\xi = 0$.

We are now in position to obtain a criterion for the linear dependence of n complex quantities. If $\alpha_1, \alpha_2, \cdots, \alpha_n$ are linearly dependent,

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = 0,$$

where not all the c's are zero. Multiplying this relation in succession by $\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_n$, we obtain the n equations

$$c_1\alpha_1\bar{\alpha}_i + c_2\alpha_2\bar{\alpha}_i + \cdots + c_n\alpha_n\bar{\alpha}_i = 0 \quad (i = 1, 2, \cdots, n).$$

In this system of homogeneous, linear equations in $c_1, c_2, \cdots, c_n$ the c's are not all zero and hence the determinant of the system must vanish. We call this determinant, which it should be noticed is a *real* scalar, the *Gramian* of $\alpha_1, \alpha_2, \cdots, \alpha_n$ and denote it by $G(\alpha_1, \alpha_2, \cdots, \alpha_n)$. Thus

$$G(\alpha_1, \alpha_2, \cdots, \alpha_n) = \begin{vmatrix} \alpha_1\bar{\alpha}_1 & \alpha_1\bar{\alpha}_2 & \cdots & \alpha_1\bar{\alpha}_n \\ \alpha_2\bar{\alpha}_1 & \alpha_2\bar{\alpha}_2 & \cdots & \alpha_2\bar{\alpha}_n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_n\bar{\alpha}_1 & \alpha_n\bar{\alpha}_2 & \cdots & \alpha_n\bar{\alpha}_n \end{vmatrix}. \tag{2}$$

The relation $G = 0$ is therefore a necessary condition for linear dependence.

It is also sufficient. For suppose that $G = 0$; then the n sets of scalars forming the rows of the Gramian are linearly dependent, and we have

$$\bar{\alpha}_i(c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n) = 0 \qquad (i = 1, 2, \cdots, n),$$

where not all of the c's vanish. We now infer from the Corollary of Theorem 1 that

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = 0,$$

which establishes the linear dependence of $\alpha_1, \alpha_2, \cdots, \alpha_n$. We have thus proved

THEOREM 2. *A necessary and sufficient condition that the complex quantities $\alpha_1, \alpha_2, \cdots, \alpha_n$ be linearly dependent is that their Gramian vanish.**

We turn now to the solution of the system (1), assuming that these equations are linearly independent, so that $G(\alpha_1, \alpha_2, \cdots, \alpha_n) \neq 0$. Every complex quantity, and therefore every solution ξ_1 of (1), can be written in the form

$$\xi_1 = c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + \cdots + c_n\bar{\alpha}_n + \bar{\eta} \tag{3}$$

where η is some complex quantity. In order that this be a solution of (1), the scalars c_i must satisfy the n relations

$$\left\{\begin{array}{l} c_1\alpha_1\bar{\alpha}_1 + c_2\alpha_1\bar{\alpha}_2 + \cdots + c_n\alpha_1\bar{\alpha}_n = -\alpha_1\bar{\eta} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ c_1\alpha_n\bar{\alpha}_1 + c_2\alpha_n\bar{\alpha}_2 + \cdots + c_n\alpha_n\bar{\alpha}_n = -\alpha_n\bar{\eta}. \end{array}\right. \tag{4}$$

Solving these equations for the c's and substituting in (3), we have

$$\xi_1 = \frac{\begin{vmatrix} \alpha_1\bar{\alpha}_1 & \alpha_1\bar{\alpha}_2 & \cdots & \alpha_1\bar{\alpha}_n & \alpha_1\bar{\eta} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_n\bar{\alpha}_1 & \alpha_n\bar{\alpha}_2 & \cdots & \alpha_n\bar{\alpha}_n & \alpha_n\bar{\eta} \\ \bar{\alpha}_1 & \bar{\alpha}_2 & \cdots & \bar{\alpha}_n & \bar{\eta} \end{vmatrix}}{G(\alpha_1, \alpha_2, \cdots, \alpha_n)}. \tag{5}$$

Every solution of (1) can therefore be expressed in this form. That, conversely, no matter what the complex quantity η may be, the expression (5) always gives a solution of (1) is seen at once by direct substitution; for if we form the product $\alpha_i\xi_1$ by multiplying the last row of the determinant in the numerator by α_i, this row becomes identical with the ith row.

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* We note in passing that $\alpha_1, \alpha_2, \cdots, \alpha_n$ are connected by the *same* linear relation that connects the rows of their Gramian, written as above.

THEOREM 3. *If the equations*

$$\alpha_1\xi = 0,\ \alpha_2\xi = 0,\ \cdots,\ \alpha_n\xi = 0,$$

are linearly independent, their general solution is given by (5), *where* η *is an arbitrary complex quantity.*

When ξ_1 vanishes, we see from (3) that $\bar{\eta}$ is linearly dependent upon $\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_n$. Conversely, if $\bar{\eta}$ is linearly dependent upon $\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_n$, the same is true of ξ_1, and hence, by Theorem 1, $\xi_1 = 0$. Now to two $\bar{\eta}$'s correspond two ξ_1's whose difference is precisely that solution of (1) which corresponds to the difference between the $\bar{\eta}$'s. Consequently two different $\bar{\eta}$'s yield the same ξ_1 when and only when their difference is linearly dependent upon $\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_n$.

If $n > k$ the equations (1) are necessarily linearly dependent, so that Theorem 3 does not apply to this case. If $n = k$ every $\bar{\eta}$ is linearly dependent on the $\bar{\alpha}$'s, so that in this case, as is well known, equations (1) have only the trivial solution zero. If $n < k$ we can find $k - n$ complex quantities $\bar{\alpha}_{n+1}, \bar{\alpha}_{n+2}, \cdots, \bar{\alpha}_k$ such that $\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_k$ are linearly independent. Then every η may be written as $C_1\bar{\alpha}_1 + C_2\bar{\alpha}_2 + \cdots + C_k\bar{\alpha}_k$; but as a change in $\bar{\eta}$ by a quantity linearly dependent upon $\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_n$ does not affect formula (5), we lose nothing in generality if we assume $\bar{\eta}$ of the form

$$\bar{\eta} = C_{n+1}\bar{\alpha}_{n+1} + \cdots + C_k\bar{\alpha}_k.$$

Thus the solution (5) contains, as it should, $k - n$ arbitrary scalars, $C_{n+1}, \cdots, C_k$, and contains them linearly and homogeneously.

A formula for the norm of ξ_1 is readily found. From (3):

$$\text{(6)} \qquad \operatorname{norm} \xi_1 = \bar{c}_1\alpha_1\xi_1 + \cdots + \bar{c}_n\alpha_n\xi_1 + \eta\xi_1 = \eta\xi_1.$$

If we form the product $\eta\xi_1$ from (5) by multiplying the last row of the determinant in the numerator by η, it is clear that

$$\text{(7)} \qquad \operatorname{norm} \xi_1 = \frac{G(\alpha_1, \alpha_2, \cdots, \alpha_n, \eta)}{G(\alpha_1, \alpha_2, \cdots, \alpha_n)}.$$

We proceed to use this relation to establish an important property of Gramians. In (7) $\alpha_1, \alpha_2, \cdots, \alpha_n, \eta$ may be regarded as $n+1$ arbitrary complex quantities; we will assume that they are linearly independent. Then $\bar{\eta}$ is clearly not a linear combination of $\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_n$, so that $\xi_1 \neq 0$ and norm $\xi_1 > 0$. Moreover this assumption entails that none of $\alpha_1, \alpha_2, \cdots, \alpha_n, \eta$ vanish, and hence the Gramian of any one, *e. g.*, $G(\alpha_1) = \alpha_1\bar{\alpha}_1$, is real and positive. Hence by giving to n in (7) in succession the values $1, 2, \cdots$, we establish by mathematical induction

THEOREM 4. *The Gramian of any number of linearly independent complex quantities is real and positive.*

3. Non-Homogeneous Linear Algebraic Equations. We come now to the system of non-homogeneous equations

$$(8)\qquad \alpha_1\xi = b_1,\ \alpha_2\xi = b_2,\ \cdots,\ \alpha_n\xi = b_n,$$

where we again assume that $\alpha_1, \alpha_2, \cdots, \alpha_n$ are linearly independent, and try to find a solution of the form

$$(9)\qquad \xi_0 = c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + \cdots + c_n\bar{\alpha}_n.$$

Substituting this in (8), we obtain n linear equations, which may be obtained from equations (4) by replacing their right hand members by $b_1, b_2, \cdots, b_n$ respectively. These can, as above, be solved for the c's by Cramer's rule, and the results substituted in (9). This gives

$$(10)\qquad \xi_0 = \frac{\begin{vmatrix} \alpha_1\bar{\alpha}_1 & \alpha_1\bar{\alpha}_2 & \cdots & \alpha_1\bar{\alpha}_n & -b_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_n\bar{\alpha}_1 & \alpha_n\bar{\alpha}_2 & \cdots & \alpha_n\bar{\alpha}_n & -b_n \\ \bar{\alpha}_1 & \bar{\alpha}_2 & \cdots & \bar{\alpha}_n & 0 \end{vmatrix}}{G(\alpha_1, \alpha_2, \cdots, \alpha_n)}.$$

That this is really a solution of (8) we see by direct substitution. For if we form the product $\alpha_i\xi_0$, the last row of the determinant in the numerator becomes

$$\alpha_i\bar{\alpha}_1,\ \alpha_i\bar{\alpha}_2,\ \cdots,\ \alpha_i\bar{\alpha}_n,\ 0;$$

and, when the ith row is subtracted from this, it appears that

$$\alpha_i\xi_0 = b_iG/G = b_i.$$

We have thus proved

THEOREM 5. *If* $\alpha_1, \alpha_2, \cdots, \alpha_n$ *are linearly independent, the equations* (8) *have one and only one solution of the form* (9), *and this is given by* (10).

The general solution of (8) is of course obtained by adding to the particular solution (10) the general solution (5) of the homogeneous equations (1); it is therefore

$$(11)\qquad \xi = \xi_0 + \xi_1 = \frac{\begin{vmatrix} \alpha_1\bar{\alpha}_1 & \alpha_1\bar{\alpha}_2 & \cdots & \alpha_1\bar{\alpha}_n & \alpha_1\bar{\eta} - b_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_n\bar{\alpha}_1 & \alpha_n\bar{\alpha}_2 & \cdots & \alpha_n\bar{\alpha}_n & \alpha_n\bar{\eta} - b_n \\ \bar{\alpha}_1 & \bar{\alpha}_2 & \cdots & \bar{\alpha}_n & \bar{\eta} \end{vmatrix}}{G(\alpha_1, \alpha_2, \cdots, \alpha_n)}.$$

The solution (10) of (8), which is characterized by being the only solution of (8) which is linearly dependent upon the $\bar{\alpha}$'s, shall be called the *principal solution* of (8). It has also another characteristic property which may be deduced as follows. From (11) we see that

$$\xi\bar{\xi} = (\xi_0 + \xi_1)(\bar{\xi}_0 + \bar{\xi}_1) = \xi_0\bar{\xi}_0 + \xi_0\bar{\xi}_1 + \xi_1\bar{\xi}_0 + \xi_1\bar{\xi}_1;$$

and from (9)

$$\xi_0\bar{\xi}_1 = c_1\bar{\alpha}_1\bar{\xi}_1 + c_2\bar{\alpha}_2\bar{\xi}_1 + \cdots + c_n\bar{\alpha}_n\bar{\xi}_1 = 0,$$

remembering that ξ_1 is a solution of equations (1). Consequently $\xi_1\bar{\xi}_0 = 0$, and

$$\text{norm } \xi = \text{norm } \xi_0 + \text{norm } \xi_1, \tag{12}$$

so that

$$\text{norm } \xi \geqq \text{norm } \xi_0,$$

the equality sign holding only when $\xi_1 = 0$, in which case $\xi = \xi_0$. Thus we have

THEOREM 6. *Among the solutions of* (8) *no other has so small a norm as the principal solution.*

To obtain a formula for norm ξ_0 we multiply the last row of the determinant in the numerator of (10) by $\bar{\xi}_0$ and simplify by use of the equations, $\bar{\alpha}_i\bar{\xi}_0 = \bar{b}_i$; thus*

$$\text{norm } \xi_0 = -\frac{\begin{vmatrix} \alpha_1\bar{\alpha}_1 & \alpha_1\bar{\alpha}_2 & \cdots & \alpha_1\bar{\alpha}_n & b_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_n\bar{\alpha}_1 & \alpha_n\bar{\alpha}_2 & \cdots & \alpha_n\bar{\alpha}_n & b_n \\ \bar{b}_1 & \bar{b}_2 & \cdots & \bar{b}_n & 0 \end{vmatrix}}{G(\alpha_1, \alpha_2, \cdots, \alpha_n)}. \tag{13}$$

Norm ξ is now given by (12).

4. System of a Finite Number of Linear Equations in an Infinite Number of Variables. We now consider a system of n equations

$$a_{i1}x_1 + a_{i2}x_2 + \cdots = 0 \qquad (i = 1, 2, \cdots, n), \tag{14}$$

where the number of unknowns $x_1, x_2, \cdots$ is infinite. For this purpose we use complex quantities with an infinite number of components. If $\alpha = (a_1, a_2, \cdots)$ is such a complex quantity, we consider the series $|a_1|^2 + |a_2| + \cdots$. If this series is convergent, we say that the complex quantity has a finite norm and define

$$\text{norm } \alpha \equiv |a_1|^2 + |a_2|^2 + \cdots, \qquad |\alpha| \equiv \sqrt{\text{norm } \alpha}.$$

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* If not all of the b's vanish, it is clear from equations (8) that $\xi_0 \neq 0$, and hence norm $\xi_0 > 0$. By means of (13) we may now prove at once the following

THEOREM. *If the Gramian of linearly independent complex quantities is bordered by scalars that do not all vanish so as to form a determinant of the type of that in* (13), *this bordered Gramian is negative.*

The sum of $\alpha = (a_1, a_2, \cdots)$ and $\beta = (b_1, b_2, \cdots)$, and the product of α by a scalar p are defined as

$$\alpha + \beta \equiv (a_1 + b_1, a_2 + b_2, \cdots), \quad p\alpha \equiv \alpha p \equiv (pa_1, pa_2, \cdots).$$

The product $\alpha\beta$ we define by the formula

$$\alpha\beta \equiv a_1b_1 + a_2b_2 + \cdots$$

whenever this series converges. When α and β have finite norms their product $\alpha\beta$ always exists, as then the series in question is absolutely convergent. For writing

$$\alpha_k = (|a_1|, |a_2|, \cdots, |a_k|), \quad \beta_k = (|b_1|, |b_2|, \cdots, |b_k|)$$

we have from Theorems 2 and 4

$$G(\alpha_k, \beta_k) = \begin{vmatrix} \alpha_k\bar{\alpha}_k & \alpha_k\bar{\beta}_k \\ \beta_k\bar{\alpha}_k & \beta_k\bar{\beta}_k \end{vmatrix} \geqq 0.$$

Hence, as $\alpha_k = \bar{\alpha}_k$ and $\beta_k = \bar{\beta}_k$,

$$(\alpha_k\beta_k)^2 \leqq \text{norm } \alpha \cdot \text{norm } \beta$$

or

$$|a_1b_1| + \cdots + |a_kb_k| \leqq |\alpha||\beta|.$$

Since this holds for all values of k, the absolute convergence of our series is established.

The distributive law, $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$, evidently holds when $\alpha\beta$ and $\alpha\gamma$ have meanings. Thus, in particular, if α and β have finite norms, we have

$$\text{norm } (\alpha + \beta) = (\alpha + \beta)(\alpha + \bar{\beta}) = \alpha\bar{\alpha} + \beta\bar{\alpha} + \alpha\bar{\beta} + \beta\bar{\beta},$$

so that if two complex quantities have finite norms their sum also has a finite norm. It is also obviously true that if a complex quantity has a finite norm it will still have a finite norm after being multiplied by a scalar. From these two facts we readily infer that if a number of complex quantities have finite norms any complex quantity linearly dependent upon them also has a finite norm.

Using the $n + 1$ complex quantities

$$\alpha_i = (a_{i1}, a_{i2}, \cdots) \qquad (i = 1, 2, \cdots, n),$$
$$\xi = (x_1, x_2, \cdots),$$

the equations (14) may be written

$$\alpha_1\xi = 0, \quad \alpha_2\xi = 0, \quad \cdots, \quad \alpha_n\xi = 0. \tag{15}$$

We place upon the coefficients α_i the restriction that they have finite norms. Then ξ is to be so determined that the series $\alpha_i\xi$ all converge to the value zero. If ξ has a finite norm the series $\alpha_i\xi$ necessarily converge, but this may also be the case when ξ has an infinite norm.

THEOREM 7. *If* ξ *satisfies the equations* (15) *and is linearly dependent on* $\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_n$, *then* $\xi = 0$.

The proof is exactly that of Theorem 1. We shall define the Gramian of a set of complex quantities of finite norm precisely as was done in § 2.

THEOREM 8. *A necessary and sufficient condition that n complex quantities of finite norm be linearly dependent is that their Gramian vanish.*

The proof is precisely that of Theorem 2.

THEOREM 9. *If equations* (15) *are linearly independent, their general solution is given by formula* (5), *where* η *is any complex quantity such that the products* $\alpha_1\bar{\eta}, \alpha_2\bar{\eta}, \cdots, \alpha_n\bar{\eta}$ *all exist.*

The proof is practically identical with that of Theorem 3. In order that the solution ξ_1 have a finite norm it is necessary and sufficient, as we see from (3), that η have a finite norm.

Here, as in § 2, it is clear that two $\bar{\eta}$'s lead to the same solution ξ_1 when and only when their difference is linearly dependent upon $\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_n$.

The requirement that η be so chosen that $\alpha_1\bar{\eta}, \alpha_2\bar{\eta}, \cdots, \alpha_n\bar{\eta}$ all exist will be fulfilled when η has a finite norm. It will, however, be fulfilled in many other cases. For example, denoting the components of α_i by $a_{i1}, a_{i2}, \cdots$, if all the a_{ij}'s are positive and a_{ij} constantly decreases and approaches zero with increasing j, we may take for η the complex quantity $(+1, -1, +1, -1, \cdots)$ whose norm is infinite.

Whenever ξ_1 has a finite norm, *i. e.*, whenever this is true of η, its norm is given by formula (7). As in §2 this formula may be now used to establish

THEOREM 10. *The Gramian of any number of linearly independent complex quantities of finite norm is real and positive.*

We now pass to the non-homogeneous equations:

$$\alpha_1\xi = b_1, \alpha_2\xi = b_2, \cdots, \alpha_n\xi = b_n, \tag{16}$$

the coefficients α_i again being assumed to have finite norms.

THEOREM 11. *If* $\alpha_1, \alpha_2, \cdots, \alpha_n$ *are linearly independent, the equations* (16) *have one and only one solution, linearly dependent upon* $\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_n$, *and this solution is given by formula* (10).

The proof is precisely that of Theorem 5. The solution in question is termed the *principal solution.* The general solution of (16) is given by formula (11), where $\bar{\eta}$ is any complex quantity whose products with $\alpha_1, \alpha_2, \cdots, \alpha_n$ exist.

THEOREM 12. *Among the solutions of* (16) *no other has so small a norm as the principal solution.*

The principal solution, being a linear combination of $\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_n$, has a finite norm. This is also true of the general solution, $\xi = \xi_0 + \xi_1$, when and only when ξ_1 has a finite norm. From here on the proof is just like that of Theorem 6.

The norm of ξ_0 is given by formula (13).*

5. Some Theorems on the Limits of Complex Quantities. We proceed to establish some properties, which will be important for us, of complex quantities with an infinite number of components.†

If α and β have finite norms, we have from Theorems 8 and 10

$$G(\alpha, \bar{\beta}) = \begin{vmatrix} \alpha\bar{\alpha} & \alpha\beta \\ \bar{\beta}\bar{\alpha} & \bar{\beta}\beta \end{vmatrix} = |\alpha|^2 |\beta|^2 - |\alpha\beta|^2 \geqq 0,$$

whence

$$|\alpha\beta| \leqq |\alpha| |\beta|. \tag{17}$$

Again, if $\gamma = \alpha + \beta$, we have, using (17) and remembering that $|\bar{\alpha}| = |\alpha|$,

$$|\gamma|^2 = (\alpha + \beta)(\bar{\alpha} + \bar{\beta}) = \alpha\bar{\alpha} + \alpha\bar{\beta} + \beta\bar{\alpha} + \beta\bar{\beta} \leqq |\alpha|^2 + 2|\alpha||\beta| + |\beta|^2,$$

$$|\gamma| \leqq |\alpha| + |\beta|. \tag{18}$$

We next lay down the following

DEFINITIONS. *If* $\alpha_n = (a_{n1}, a_{n2}, \cdots)$, $\alpha = (a_1, a_2, \cdots)$, *we say that* α_n *converges to* α *as* n *becomes infinite when*

$$\lim_{n=\infty} a_{ni} = a_i \qquad (i = 1, 2, \cdots),$$

and write

$$\lim_{n=\infty} \alpha_n = \alpha.$$

We say that α_n *has strong convergence toward* α *when, for all values of* n *greater than a certain number,* $\alpha - \alpha_n$ *has a finite norm, and*

$$\lim_{n=\infty} |\alpha - \alpha_n| = 0,$$

and write, using Schmidt's notation,

$$\underline{\underline{\lim_{n=\infty}}}\, \alpha_n = \alpha.$$

Strong convergence implies convergence. For if $\lim_{n=\infty} |\alpha - \alpha_n| = 0$, there exists, for every positive ϵ, an integer N such that

$$\sum_{i=1}^{\infty} |a_i - a_{ni}|^2 < \epsilon^2, \qquad n > N, \tag{19}$$

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* The theorem regarding bordered Gramians, stated in the footnote to formula (13), may now be generalized so as to apply to the Gramians of complex quantities with finite norms.

† Due to E. Schmidt, l. c., §§ 1–4. See also Kowalewski, l. c., § 165.

so that

$$(20) \qquad | a_i - a_{ni} | < \epsilon, \qquad \begin{Bmatrix} (i = 1, 2, \cdots), \\ n > N, \end{Bmatrix}$$

or

$$\lim_{n=\infty} a_{ni} = a_i \qquad (i = 1, 2, \cdots).*$$

If $\underline{\lim}_{n=\infty} \alpha_n = \alpha$ and α_n has a finite norm when n is greater than a certain number, then α will have a finite norm. For (19) states that when $n > N$, $\alpha - \alpha_n$ has a finite norm; consequently the sum of α_n and $\alpha - \alpha_n$ has a finite norm.

Again, if $\underline{\lim}_{n=\infty}\alpha_n = \alpha$, $\underline{\lim}_{n=\infty}\beta_n = \beta$, then

$$(21) \qquad \underline{\lim}_{n=\infty}(\alpha_n + \beta_n) = \alpha + \beta;$$

for we have seen that when $n > N$, $\alpha - \alpha_n$ and $\beta - \beta_n$ have finite norms, and hence from (18) we have

$$| \alpha + \beta - \alpha_n - \beta_n | \leqq | \alpha - \alpha_n | + | \beta - \beta_n |.$$

Furthermore if α_n, β_n have finite norms when $n > N$, so that α, β have finite norms,

$$(22) \qquad \lim_{n=\infty} \alpha_n\beta_n = \alpha\beta;$$

for when $n > N$, we have, using (17) and (18),

$$\begin{aligned} | \alpha\beta - \alpha_n\beta_n | &= | (\alpha - \alpha_n)\beta + (\beta - \beta_n)\alpha - (\alpha - \alpha_n)(\beta - \beta_n) | \\ &\leqq | \alpha - \alpha_n | | \beta | + | \beta - \beta_n | | \alpha | + | \alpha - \alpha_n | | \beta - \beta_n |. \end{aligned}$$

Important special cases of (22) are

$$(23) \qquad \lim_{n=\infty} \alpha_n\beta = \alpha\beta;$$

$$(24) \qquad \lim_{n=\infty} \text{norm } \alpha_n = \text{norm } \alpha.$$

THEOREM 13. *A necessary and sufficient condition that* $\underline{\lim}_{n=\infty} \alpha_n$ *exist is that, when n and m are any integers greater than a certain number,* $\alpha_n - \alpha_m$ *have a finite norm, and that to every positive* ϵ *there correspond an integer N such that*

$$(25) \qquad | \alpha_n - \alpha_m | < \epsilon, \qquad m, n > N.$$

The condition is necessary; for if $\underline{\lim}_{n=\infty} \alpha_n = \alpha$, $| \alpha - \alpha_n | < \frac{1}{2}\epsilon$ when $n > N$. Hence when $m, n > N$

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* We say that α_n converges *uniformly* toward α when for every positive ϵ there exists an N such that (20) is true. It is clear from the above that strong convergence implies uniform convergence, and uniform convergence implies convergence; but these implications do not hold in the reverse order.

$$| \alpha_n - \alpha_m | = | \alpha_n - \alpha + \alpha - \alpha_m | \leqq | \alpha_n - \alpha | + | \alpha - \alpha_m | < \epsilon.$$

To show the sufficiency of the condition we first observe that if (25) holds,

$$\sum_{k=1}^{\infty} | a_{nk} - a_{mk} |^2 < \epsilon^2, \qquad m, n > N,$$

and hence

$$|a_{nk} - a_{mk}| < \epsilon, \qquad \left\{ \begin{matrix} (k = 1, 2, \cdots), \\ m, n > N. \end{matrix} \right\}$$

This shows that $\lim_{n=\infty} a_{nk}$ exists; denote it by a_k. Then as

$$\sum_{k=1}^{p} | a_{nk} - a_{mk} |^2 < \epsilon^2, \qquad m, n > N,$$

$$\lim_{n=\infty} \sum_{k=1}^{p} | a_{nk} - a_{mk} |^2 = \sum_{k=1}^{p} | a_k - a_{mk} |^2 \leqq \epsilon^2, \qquad m > N.$$

As this holds for every p, we have

$$\sum_{k=1}^{\infty} | a_k - a_{mk} |^2 \leqq \epsilon^2, \qquad m > N;$$

or, upon writing $\alpha = (a_1, a_2, \cdots)$,

$$\lim_{m=\infty} | \alpha - \alpha_m | = 0$$

as we wished to prove.

COROLLARY. *When condition (25) is fulfilled and α_n is always of finite norm, α is also of finite norm.*

DEFINITION. *The infinite series of complex quantities $\alpha_1 + \alpha_2 + \cdots$ is said to converge strongly to a complex quantity σ when σ_n converges strongly to σ, where $\sigma_n = \alpha_1 + \cdots + \alpha_n$.*

From Theorem 13 we see that a necessary and sufficient condition for the strong convergence of the above series is that after a certain point the terms of the series all have finite norms and that, to every positive ϵ, there correspond an integer N such that

$$(26) \qquad | \sigma_n - \sigma_m | = | \alpha_{m+1} + \alpha_{m+2} + \cdots + \alpha_n | < \epsilon, \qquad m, n > N.$$

DEFINITION. *Two complex quantities α, β, are said to be orthogonal if $\alpha\bar{\beta}$, and hence also $\bar{\alpha}\beta$, is zero.*

If the α's have finite norms and are mutually orthogonal, we may, by squaring (26), readily reduce it to the form

$$| \alpha_{m+1} |^2 + | \alpha_{m+2} |^2 + \cdots + | \alpha_n |^2 < \epsilon^2, \qquad m, n > N.$$

This being precisely a necessary and sufficient condition that the series $| \alpha_1 |^2 + | \alpha_2 |^2 + \cdots$ converge, we have proved

THEOREM 14. *A series of mutually orthogonal complex quantities of* *finite norm is strongly convergent when and only when the series of their norms converges.**

Furthermore as $|\sigma_n|^2 = |\alpha_1|^2 + |\alpha_2|^2 + \cdots + |\alpha_n|^2$, we infer from (24) the

COROLLARY. *If the conditions of Theorem 14 are fulfilled, the norm of the series is equal to the series of the norms of the terms.*

6. System of an Infinite Number of Linear Equations in an Infinite Number of Variables. We are now in position to consider the infinite system of homogeneous equations in an infinite number of variables

$$\alpha_1\xi = 0, \quad \alpha_2\xi = 0, \cdots, \tag{27}$$

where

$$\alpha_i = (a_{i1}, a_{i2}, \cdots) \qquad (i = 1, 2, \cdots),$$

$$\xi = (x_1, x_2, \cdots).$$

We assume that all the coefficients α_i have finite norms and none of them are linearly dependent. The general solution, $\xi_1^{(n)}$, of the first n of these equations is given by formula (5)

$$\xi_1^{(n)} = \sum_{i=1}^{n} c_i^{(n)}\bar{\alpha}_i + \bar{\eta} = \frac{\begin{vmatrix} \alpha_1\bar{\alpha}_1 & \alpha_1\bar{\alpha}_2 & \cdots & \alpha_1\bar{\alpha}_n & \alpha_1\bar{\eta} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_n\bar{\alpha}_1 & \alpha_n\bar{\alpha}_2 & \cdots & \alpha_n\bar{\alpha}_n & \alpha_n\bar{\eta} \\ \bar{\alpha}_1 & \bar{\alpha}_2 & \cdots & \bar{\alpha}_n & \bar{\eta} \end{vmatrix}}{G(\alpha_1, \alpha_2, \cdots, \alpha_n)}. \tag{28}$$

Here $(c_1^{(n)}, c_2^{(n)}, \cdots, c_n^{(n)})$ is a solution of equations (4).

We wish to show that $\xi_1^{(n)}$ converges strongly to a limit as $n = \infty$; and to this end we proceed to throw it into the form

$$\xi_1^{(n)} = \xi_1^{(1)} + (\xi_1^{(2)} - \xi_1^{(1)}) + \cdots + (\xi_1^{(n)} - \xi_1^{(n-1)}).$$

If we write

$$c_i^{(n)} - c_i^{(n-1)} = z_i^{(n)} \qquad (i = 1, 2, \cdots, n-1),$$

$$c_n^{(n)} = z_n^{(n)},$$

and subtract from the first $n - 1$ equations (4) the similar equations satisfied by $(c_1^{(n-1)}, \cdots, c_{n-1}^{(n-1)})$, we find that the z's satisfy the $n - 1$ *homogeneous* equations

$$\left\{\begin{array}{l} \alpha_1\bar{\alpha}_1 z_1^{(n)} + \alpha_1\bar{\alpha}_2 z_2^{(n)} + \cdots + \alpha_1\bar{\alpha}_n z_n^{(n)} = 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \alpha_{n-1}\bar{\alpha}_1 z_1^{(n)} + \alpha_{n-1}\bar{\alpha}_2 z_2^{(n)} + \cdots + \alpha_{n-1}\bar{\alpha}_n z_n^{(n)} = 0. \end{array}\right.$$

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* The proof above establishes the more general theorem in which the condition of orthogonality is replaced by the condition $\alpha_i\bar{\alpha}_j + \alpha_j\bar{\alpha}_i = 0$ or (real part of $\alpha_i\,\bar{\alpha}_j$) $= 0$ when $i \neq j$, $i, j = 1, 2, \cdots$.

Moreover we have

$$\xi_1^{(n)} - \xi_1^{(n-1)} = z_1^{(n)}\bar{\alpha}_1 + z_2^{(n)}\bar{\alpha}_2 + \cdots + z_n^{(n)}\bar{\alpha}_n \qquad (n = 2, 3, \cdots).$$

Solving the homogeneous equations for the z's and substituting in the last equation, we have

$$\xi_1^{(n)} - \xi_1^{(n-1)} = k_n\varphi_n, \tag{29}$$

where k_n is an undetermined scalar and

$$\varphi_n \equiv \begin{vmatrix} \alpha_1\bar{\alpha}_1 & \alpha_1\bar{\alpha}_2 & \cdots & \alpha_1\alpha_n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_{n-1}\bar{\alpha}_1 & \alpha_{n-1}\bar{\alpha}_2 & \cdots & \alpha_{n-1}\bar{\alpha}_n \\ \bar{\alpha}_1 & \bar{\alpha}_2 & \cdots & \bar{\alpha}_n \end{vmatrix} \qquad (n = 2, 3, \cdots).$$

Multiplying both sides of (29) by α_n and using (28), we find $-H_n/G_{n-1} = k_nG_n$, where, for brevity, we have written

$$H_n = \begin{vmatrix} \alpha_1\bar{\alpha}_1 & \alpha_1\bar{\alpha}_2 & \cdots & \alpha_1\bar{\alpha}_{n-1} & \alpha_1\bar{\eta} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_{n-1}\bar{\alpha}_1 & \alpha_{n-1}\bar{\alpha}_2 & \cdots & \alpha_{n-1}\bar{\alpha}_{n-1} & \alpha_{n-1}\bar{\eta} \\ \alpha_n\bar{\alpha}_1 & \alpha_n\bar{\alpha}_2 & \cdots & \alpha_n\bar{\alpha}_{n-1} & \alpha_n\bar{\eta} \end{vmatrix} \qquad (n = 2, 3, \cdots),$$

$$G_n = G(\alpha_1, \cdots, \alpha_n).$$

Therefore

$$\xi_1^{(n)} - \xi_1^{(n-1)} = -\frac{H_n}{G_{n-1}G_n}\varphi_n;$$

and as $\xi_1^{(1)} = \bar{\eta} - (\alpha_1\bar{\eta}/\alpha_1\bar{\alpha}_1)\bar{\alpha}_1$, we have, if we set $\varphi_1 = \bar{\alpha}_1$, $G_0 = 1$, $H_1 = \alpha_1\bar{\eta}$,

$$\xi_1^{(m)} = \bar{\eta} - \sum_{n=1}^{n=m}\frac{H_n}{G_{n-1}G_n}\varphi_n. \tag{30}$$

If η, and hence $\xi_1^{(m)}$, has a finite norm, we see from (6) that norm $\xi_1^{(m)} = \eta\xi_1^{(m)}$. Assuming, then, that this is the case, we have, since $\varphi_n\eta = \bar{H}_n$,

$$\text{norm } \xi_1^{(m)} = |\eta|^2 - \sum_{n=1}^{n=m}\frac{|H_n|^2}{G_{n-1}G_n}. \tag{31}$$

The series of positive or zero terms

$$\sum_{n=1}^{\infty}\frac{|H_n|^2}{G_{n-1}G_n} \tag{32}$$

is therefore convergent for every η of finite norm since the sum of its first m terms is by (31) not greater than $|\eta|^2$.

We next note that the terms of the series of complex quantities of finite norm

$$\sum_{n=1}^{\infty} \frac{H_n}{G_{n-1}G_n} \varphi_n \tag{33}$$

are mutually orthogonal; for as

$$\varphi_n \alpha_i = 0 \qquad (i = 1, 2, \cdots, n-1),$$

we have

$$\varphi_n \bar{\varphi}_m = 0, \qquad m < n.$$

By Theorem (14) the series (33) will converge strongly if the series of the norms of its terms converges. If we use the relation

$$\varphi_n \bar{\varphi}_n = \varphi_n \alpha_n G_{n-1} = G_n G_{n-1},$$

this series of norms proves to be precisely (32), which we have just shown to be convergent when η is of finite norm. Hence series (33) converges strongly when η has a finite norm, as does likewise the series

$$\xi_1 \equiv \lim_{m=\infty} \xi_1^{(m)} = \bar{\eta} - \sum_{n=1}^{\infty} \frac{H_n}{G_{n-1}G_n} \varphi_n = \bar{\eta} - \sum_{n=1}^{\infty} \frac{\bar{\eta}\bar{\varphi}_n}{\phi_n \bar{\varphi}_n} \varphi_n. \tag{34}$$

ξ_1 is a solution of equations (27) having a finite norm. For consider any one of these equations, say $\alpha_k \xi = 0$; since

$$\alpha_k \xi_1^{(m)} = 0 \qquad (m = k, k+1, \cdots),$$

we have from (23)

$$\lim_{m=\infty} (\alpha_k \xi_1^{(m)}) = \alpha_k \xi_1 = 0.$$

That ξ_1 is of finite norm follows from the fact that $\xi_1^{(m)}$ is always of finite norm and converges *strongly* towards ξ_1.

Conversely, if ξ_1 is any solution of equations (27), we may obtain it by letting $\bar{\eta} = \xi_1$ in the formula (34), for then all the terms after the first vanish. Thus we have proved

THEOREM 15. *If η is a complex quantity of finite norm, $\xi_1^{(n)}$, given by formula* (28), *approaches a limiting complex quantity of finite norm as n becomes infinite, and this limit, ξ_1, is a solution of the equations* (27).

Conversely, every solution of (27), *whether of finite norm or not, can be obtained by properly choosing η in* (34).

From formulas (24) and (31) we have

$$\text{norm } \xi_1 = \lim_{m=\infty} \text{norm } \xi_1^{(m)} = |\eta|^2 - \sum_{m=1}^{\infty} \frac{|H_n|^2}{G_{n-1}G_n} \tag{35}$$

whenever η is of finite norm. Referring to (7), we see that this may also be written as

$$\text{norm } \xi_1 = \lim_{n=\infty} \frac{G(\alpha_1, \alpha_2, \cdots, \alpha_n, \eta)}{G(\alpha_1, \alpha_2, \cdots, \alpha_n)}. \tag{36}$$

We turn now to the non-homogeneous equations

$$\alpha_1\xi = b_1, \quad \alpha_2\xi = b_2, \quad \cdots, \tag{37}$$

where we again assume that all the coefficients α_i have finite norms and none of them are linearly dependent. The principal solution of the first n of these equations, which we will denote by $\xi_0^{(n)}$, is given by formula (10)

$$\xi_0^{(n)} = \sum_{i=1}^{n} c_i^{(n)}\bar{\alpha}_i = \frac{\begin{vmatrix} \alpha_1\bar{\alpha}_1 & \alpha_1\bar{\alpha}_2 & \cdots & \alpha_1\bar{\alpha}_n & -b_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_n\bar{\alpha}_1 & \alpha_n\bar{\alpha}_2 & \cdots & \alpha_n\bar{\alpha}_n & -b_n \\ \bar{\alpha}_1 & \bar{\alpha}_2 & \cdots & \bar{\alpha}_n & 0 \end{vmatrix}}{G(\alpha_1, \alpha_2, \cdots, \alpha_n)}. \tag{38}$$

Here $(c_1^{(n)}, c_2^{(n)}, \cdots, c_n^{(n)})$ is a solution of the equations obtained from (4) by replacing their right-hand members, $-\alpha_1\bar{\eta}$, $-\alpha_2\bar{\eta}$, $\cdots$, $-\alpha_n\bar{\eta}$ by $b_1, b_2, \cdots, b_n$ respectively. A consideration of the process by which $\xi_1^{(n)} - \xi_1^{(n-1)}$ was obtained shows that we may obtain $\xi_0^{(n)} - \xi_0^{(n-1)}$ from this expression by replacing $-\alpha_1\bar{\eta}$, $-\alpha_2\bar{\eta}$, $\cdots$, $-\alpha_n\bar{\eta}$ by $b_1, b_2, \cdots, b_n$ respectively; consequently in place of $-H_n$ we must now introduce the determinant

$$B_1 = b_1, \quad B_n \equiv \begin{vmatrix} \alpha_1\bar{\alpha}_1 & \alpha_1\bar{\alpha}_2 & \cdots & \alpha_1\bar{\alpha}_{n-1} & b_1 \\ \alpha_2\bar{\alpha}_1 & \alpha_2\bar{\alpha}_2 & \cdots & \alpha_2\bar{\alpha}_{n-1} & b_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_n\bar{\alpha}_1 & \alpha_n\bar{\alpha}_2 & \cdots & \alpha_n\bar{\alpha}_{n-1} & b_n \end{vmatrix} \quad (n = 2, 3, \cdots)$$

and we obtain

$$\xi_0^{(n)} - \xi_0^{(n-1)} = \frac{B_n}{G_{n-1}G_n}\varphi_n.$$

As

$$\xi_0^{(1)} = \frac{b_1}{\alpha_1\bar{\alpha}_1}\bar{\alpha}_1,$$

$$\xi_0^{(m)} = \sum_{n=1}^{n=m} \frac{B_n}{G_{n-1}G_n}\varphi_n. \tag{39}$$

We are thus led to consider the series

$$\sum_{n=1}^{\infty} \frac{B_n}{G_{n-1}G_n}\varphi_n \tag{40}$$

whose terms are mutually orthogonal complex quantities of finite norm—as we know from the previously established properties of φ_n. By Theorem 14 this series will be strongly convergent when and only when the series of the norms of its terms

$$\sum_{n=1}^{\infty} \frac{|B_n|^2}{G_{n-1}G_n} \tag{41}$$

converges. Thus when series (41) converges we have

$$\xi_0 \equiv \lim_{\overline{m=\infty}} \xi_0^{(m)} = \sum_{n=1}^{\infty} \frac{B_n}{G_{n-1}G_n} \varphi_n \tag{42}$$

and an argument similar to that which follows (34) shows that ξ_0 is a solution of equations (37) having a finite norm. Now if equations (37) have *any* solution, ξ, of finite norm, then, as $\xi_0^{(m)}$ is the solution of least norm of the first m of these equations,

$$\text{norm } \xi_0^{(m)} \leqq \text{norm } \xi;$$

and since norm $\xi_0^{(m)}$ proves to be precisely the sum of the first m terms of (41), the convergence of this series is established. Thus we have proved

THEOREM 16. *A necessary and sufficient condition that equations* (37) *have a solution of finite norm is that the series* (41) *converge. When this is the case,* $\xi_0^{(n)}$, *given by formula* (38), *approaches strongly a limiting complex quantity of finite norm as n becomes infinite, and this limit,* ξ_0, *is a solution of the equations.*

ξ_0 is termed the *principal solution* of (37). We may form the general solution by adding to the particular solution ξ_0 the general solution ξ_1 of equations (27):

$$\xi = \xi_0 + \xi_1 = \lim_{\overline{n=\infty}} \frac{\begin{vmatrix} \alpha_1\bar{\alpha}_1 & \alpha_1\bar{\alpha}_2 & \cdots & \alpha_1\bar{\alpha}_n & \alpha_1\bar{\eta} - b_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_n\bar{\alpha}_1 & \alpha_n\bar{\alpha}_2 & \cdots & \alpha_n\bar{\alpha}_n & \alpha_n\bar{\eta} - b_n \\ \bar{\alpha}_1 & \bar{\alpha}_2 & \cdots & \bar{\alpha}_n & \bar{\eta} \end{vmatrix}}{G(\alpha_1, \alpha_2, \cdots, \alpha_n)}. \tag{43}$$

From the Corollary to Theorem 14 we have

$$\text{norm } \xi_0 = \sum_{n=1}^{\infty} \frac{|B_n|^2}{G_{n-1}G_n},$$

or, referring to (13),

$$\text{norm } \xi_0 = \lim_{n=\infty} - \frac{\begin{vmatrix} \alpha_1\bar{\alpha}_1 & \alpha_1\bar{\alpha}_2 & \cdots & \alpha_1\bar{\alpha}_n & b_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_n\bar{\alpha}_1 & \alpha_n\bar{\alpha}_2 & \cdots & \alpha_n\bar{\alpha}_n & b_n \\ \bar{b}_1 & \bar{b}_2 & \cdots & \bar{b}_n & 0 \end{vmatrix}}{G(\alpha_1, \alpha_2, \cdots, \alpha_n)}. \tag{44}$$

If ξ_1 has a finite norm, the same is true of ξ, and

$$\text{norm } \xi = \text{norm } \xi_0 + \text{norm } \xi_1,$$

for from (38) $\xi_0^{(n)}\bar{\xi}_1 = 0$, so that upon applying (23), $\xi_0\bar{\xi}_1 = \xi_1\bar{\xi}_0 = 0$. Consequently

$$\text{norm } \xi \geqq \text{norm } \xi_0$$

the sign of equality holding only when $\xi_1 = 0$, in which case $\xi = \xi_0$. Thus we have proved

THEOREM 17. *Among the solutions of* (37) *no other has so small a norm as the principal solution.*

7. Some further facts.—The general solution ξ_1 of the homogeneous equations (27) is a function of the complex parameter η

$$\xi_1 = \psi(\eta).$$

A glance at (28) shows us at once that ψ is, in an extended sense, a *linear* function; that is

THEOREM 18. *If* $\eta', \eta'', \cdots, \eta^{[k]}$ *are complex quantities with finite norms and* $c_1, \cdots, c_k$ *are scalars, then*

$$\psi(c_1\eta' + \cdots + c_k\eta^{[k]}) = c_1\psi(\eta') + \cdots + c_k\psi(\eta^{[k]}).$$

A further important fact is that ψ has *strong continuity* for every value of η with finite norm; that is

THEOREM 19. *If* η' *has a finite norm, then as* η *approaches* η' *strongly,* $\psi(\eta)$ *approaches* $\psi(\eta')$ *strongly.*

To prove this, we derive from Theorem 18 and from (35) the relation

$$\text{norm } [\psi(\eta') - \psi(\eta)] = \text{norm } \psi(\eta' - \eta) \leqq \text{norm } (\eta' - \eta),$$

from which our theorem follows at once.

Let us now denote the components of η by $y_1, y_2, \cdots\cdot$, and the complex quantity whose first n components are $y_1, \cdots, y_n$ while all its subsequent components are zero by η_n. Then, if η is of finite norm,

$$\lim_{n=\infty} \eta_n = \eta. \tag{45}$$

For norm $(\eta - \eta_n) = |y_{n+1}|^2 + |y_{n+2}|^2 + \cdots\cdot$, and, this being the remainder of a convergent series, approaches zero as n becomes infinite.

Let us denote by ϵ_i the complex quantity whose ith component is 1 while all its other components are zero. Then

$$\psi(\epsilon_i) = \lim_{n=\infty} \frac{\begin{vmatrix} \alpha_1\bar{\alpha}_1 & \cdots & \alpha_1\bar{\alpha}_n & a_{1i} \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_n\bar{\alpha}_1 & \cdots & \alpha_n\bar{\alpha}_n & a_{ni} \\ \bar{\alpha}_1 & \cdots & \bar{\alpha}_n & \epsilon_i \end{vmatrix}}{G(\alpha_1, \cdots, \alpha_n)}.$$

THEOREM 20. *A necessary and sufficient condition that the homogeneous system* (27) *have no solution of finite norm except zero is that all the quantities* $\psi(\epsilon_i)$ *be zero.*

That this is a necessary condition is obvious. To prove it sufficient assume $\psi(\epsilon_i) = 0 \ (i = 1, 2, \cdots)$. By Theorem 18, $\psi(\eta) = 0$ whenever η has only a finite number of components different from zero. But, by (45), every η of finite norm is the strong limit of such a set of η's. Consequently, by Theorem 19, $\psi(\eta) = 0$ for every η of finite norm, as was to be proved.

We have expressed the solutions ξ_1 and ξ_0 as well as their norms, as the limit of the ratio of two determinants of order $n + 1$ and n as n becomes infinite. We proceed to inquire under what conditions the individual determinants, and not merely their ratios, converge. In all cases the denominator determinant is $G(\alpha_1, \cdots, \alpha_n)$, and if this Gramian converges as n becomes infinite, the determinants in the numerators will likewise converge. Thus we have merely to consider the convergence of $G(\alpha_1, \cdots, \alpha_n)$ as n becomes infinite, or, as we phrase it, the convergence of the *infinite Gramian*, $G(\alpha_1, \alpha_2, \cdots)$.

THEOREM 21. *A sufficient condition for the convergence of the infinite Gramian of the complex quantities* $\alpha_1, \alpha_2, \cdots$ *which have finite norms is that the infinite product* $\prod_{i=1}^{\infty} |\alpha_i|^2$ *diverge to zero or converge.*

Consider the set of complex quantities $\beta_i = \alpha_i/|\alpha_i|$ whose norms are all unity. We have, then,

$$G(\alpha_1, \cdots, \alpha_n) = G(\beta_1, \cdots, \beta_n) \prod_{i=1}^{n} |\alpha_i|^2. \tag{46}$$

Now

$$G(\beta_1, \cdots, \beta_n) = \begin{vmatrix} \beta_1\bar{\beta}_1 & \cdots & \beta_1\bar{\beta}_{n-1} & \beta_1\bar{\beta}_n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \beta_{n-1}\bar{\beta}_1 & \cdots & \beta_{n-1}\bar{\beta}_{n-1} & \beta_{n-1}\bar{\beta}_n \\ \beta_n\bar{\beta}_1 & \cdots & \beta_n\bar{\beta}_{n-1} & 0 \end{vmatrix} + |\beta_n|^2 G(\beta_1, \cdots, \beta_{n-1}).$$

The first term on the right is a bordered Gramian of the form of the numerator of (13) and is therefore negative or zero (see footnote at the end of § 3). Consequently

$$G(\beta_1, \cdots, \beta_n) \leqq G(\beta_1, \cdots, \beta_{n-1});$$

and since $G(\beta_1, \cdots, \beta_n)$ is never negative, $\lim_{n=\infty} G(\beta_1, \cdots, \beta_n)$ exists. Thus when $\prod_{i=1}^{\infty} |\alpha_i|^2$ diverges to zero or converges, we have from (46) that $G(\alpha_1, \alpha_2, \cdots)$ converges, as we wished to prove.

COROLLARY 1. *If* $G(\beta_1, \beta_2, \cdots) \neq 0$ *the condition that* $\prod_{i=1}^{\infty} |\alpha_i|^2$ *diverge to zero or converge is also necessary for the convergence of* $G(\alpha_1, \alpha_2, \cdots)$.

Corollary 2. *If* $\prod_{i=1}^{\infty} | \alpha_i |^2 = 0$, *then* $G(\alpha_1, \alpha_2, \cdots) = 0$.

We also note that $G(\alpha_1, \alpha_2, \cdots) = 0$ when any of the complex quantities α_i are linearly dependent.

From Theorem 21 we now see that the determinants occurring in the expressions for $\xi_1^{(n)}$ and $\xi_0^{(n)}$ (and for their norms) will converge as $n = \infty$ if, at the start, the equations (27) and (37) respectively are divided through by scalars so as to make the norms of all the α's $\leqq 1$. If, when this is done, $G(\alpha_1, \alpha_2, \cdots) \neq 0$, the formulæ for ξ_1 and ξ_0 furnish solutions for these infinite systems of equations in terms of infinite determinants, properly so called. Of course the last row and column of the numerator determinants must then be written as first row and column.

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